

Casimir Energy of the Universe and New Regularization of Higher Dimensional Quantum Field Theories

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Abstract. Casimir energy is calculated for the 5D electromagnetism and 5D scalar theory in the *warped* geometry. It is compared with the flat case. A new regularization, called *sphere lattice regularization*, is taken. In the integration over the 5D space, we introduce two boundary curves (IR-surface and UV-surface) based on the *minimal area principle*. It is a *direct* realization of the geometrical approach to the *renormalization group*. The regularized configuration is *closed-string like*. We do *not* take the KK-expansion approach. Instead, the position/momentum propagator is exploited, combined with the *heat-kernel method*. All expressions are closed-form (not KK-expanded form). The *generalized* P/M propagators are introduced. We numerically evaluate Λ (4D UV-cutoff), ω (5D bulk curvature, warp parameter) and T (extra space IR parameter) dependence of the Casimir energy. We present two *new ideas* in order to define the 5D QFT: 1) the summation (integral) region over the 5D space is *restricted* by two minimal surfaces (IR-surface, UV-surface) ; or 2) we introduce a *weight function* and require the dominant contribution, in the summation, is given by the *minimal surface*. Based on these, 5D Casimir energy is *finitely* obtained after the *proper renormalization procedure*. The *warp parameter* ω suffers from the *renormalization effect*. The IR parameter T does not. We examine the meaning of the weight function and finally reach a *new definition* of the Casimir energy where the *4D momenta(or coordinates)* are *quantized* with the extra coordinate as the Euclidean time (inverse temperature). We examine the cosmological constant problem and present an answer at the end. Dirac's large number naturally appears.

1. Introduction

In the dawn of the quantum theory, the *divergence* problem of the specific heat of the radiation cavity was the biggest one (the problem of the blackbody radiation). It is historically so famous that the difficulty was solved by Planck's idea that the energy is quantized. In other words, the phase space of the photon field dynamics is not continuous but has the "cell" or "lattice" structure with the unit area ($\Delta x \cdot \Delta p$) of the size $2\pi\hbar$ (Planck constant). The radiation energy is composed of two parts, E_{Cas} and E_β :

$$E_{AdEM} = E_{Cas} + E_\beta, \quad E_{Cas} = \sum_{m_x, m_y, n \in \mathbf{Z}} \tilde{\omega}_{m_x m_y n}, \quad E_\beta = 2 \sum_{m_x, m_y, n \in \mathbf{Z}} \frac{\tilde{\omega}_{m_x m_y n}}{e^{\beta \tilde{\omega}_{m_x m_y n}} - 1},$$

$$\tilde{\omega}_{m_x m_y n}^2 = (m_x \frac{\pi}{L})^2 + (m_y \frac{\pi}{L})^2 + (n \frac{\pi}{l})^2, \quad l \ll L, \quad (1)$$

where the parameter β is the inverse temperature, l is the separation length between two perfectly-conducting plates, and L is the IR regularization parameter of the plate-size. The second part E_β is, essentially, Planck's radiation formula. The first one E_{Cas} is the vacuum energy of the radiation field, that is, the Casimir energy. It is a very delicate quantity. The quantity is formally *divergent*, hence it must be defined with careful *regularization*. $E_{Cas}/(2L)^2$ does depend only on the *boudary* parameter l . The quantity is a quantum effect and , at the same time, depends on the *global* (macro) parameter l .

$$\frac{E_{Cas}}{(2L)^2} = \frac{\pi^2}{(2l)^3} \frac{B_4}{4!} = -\frac{\pi^2}{720} \frac{1}{(2l)^3} \quad , \quad B_4(\text{the fourth Bernoulli number}) = -\frac{1}{30} \quad , \quad (2)$$

In Fig.1, Planck's radiation spectrum distribution is shown. Introducing the axis of the

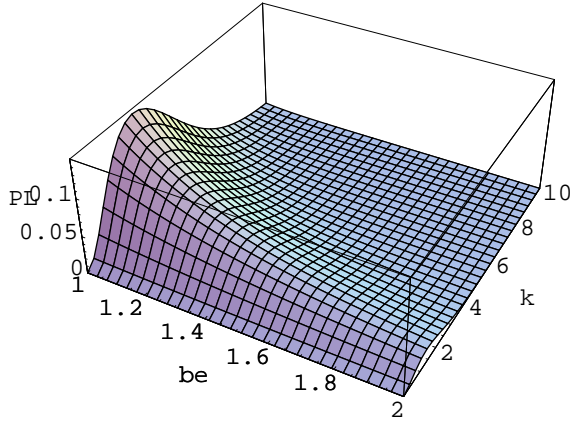


Figure 1. Graph of Planck's radiation formula. $\mathcal{P}(\beta, k) = \frac{1}{(ch)^3} \frac{1}{\pi^2} k^3 / (e^{\beta k} - 1)$ ($1 \leq \beta \leq 2$, $0.01 \leq k \leq 10$).

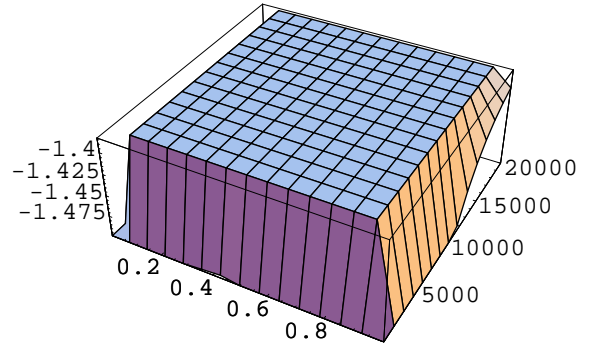


Figure 2. Behavior of $\ln |\frac{1}{2} \mathcal{F}^-(\tilde{k}, z)| = \ln |\tilde{k} G_k^-(z, z) / (\omega z)^3|$. $\omega = 10^4$, $T = 1$, $\Lambda = 2 \times 10^4$. $1.0001/\omega \leq z \leq 0.9999/T$. $\Lambda T/\omega \leq \tilde{k} \leq \Lambda$. Note $\ln |(1/2) \times (1/2)| \approx -1.39$.

inverse temperature(β), besides the photon energy or frequency (k), it is shown stereographically. Although we will examine the 5D version of the zero-point part (the Casimir energy), the calculated quantities in this paper are much more related to this Planck's formula. We see, near the β -axis, a sharply-rising surface, which is the Rayleigh-Jeans region (the energy density is proportional to the *square* of the photon frequency). The *damping* region in high k is the Wien's region.¹ The ridge (the line of peaks at each β) forms the *hyperbolic* curve (Wien's displacement law). When we will, in this paper, deal with the energy distribution over the 4D momentum and the extra-coordinate, we will see the similar behavior (although top and bottom appear in the opposite way).

In the quest for the unified theory, the higher dimensional (HD) approach is a fascinating one from the geometrical point. Historically the initial successful one is the Kaluza-Klein model[1, 2], which unifies the photon, graviton and dilaton from the 5D space-time approach. The HD theories , however, generally have the serious defect as the quantum field theory(QFT) : unrenormalizability. The HD quantum field theories, at present, are not defined within the QFT. One can take the standpoint that the more fundamental formulation, such as the string theory

¹ We recall that the old problem of the *divergent* specific heat was solved by the Wien's formula. This fact strongly supports the present idea of introducing the *weight function* (see Sec.6).

and D-brane theory, can solve the problem. In the present paper, we have the *new standpoint* that the HD theories should be defined by themselves within the QFT. In order to escape the dimension requirement $D=10$ or 26 from the quantum consistency (anomaly cancellation)[3], we treat the gravitational (metric) field only as the background one. This does *not* mean the space-time is not quantized. See later discussions (Sec.7). We present a way to define 5D quantum field theory through the analysis of the Casimir energy of 5D electromagnetism.

In 1983, the Casimir energy in the Kaluza-Klein theory was calculated by Appelquist and Chodos[4]. They took the cut-off (Λ) regularization and found the quintic (Λ^5) divergence and the finite term. The divergent term shows the *unrenormalizability* of the 5D theory, but the finite term looks meaningful ² and, in fact, is widely regarded as the right vacuum energy which shows *contraction* of the extra axis.

In the development of the string and D-brane theories, a new approach to the renormalization group was found. It is called *holographic renormalization* [6, 7, 8, 9, 10, 11]. We regard the renormalization flow as a curve in the bulk (HD space). The flow goes along the extra axis. The curve is derived as a dynamical equation such as Hamilton-Jacobi equation. It originated from the AdS/CFT correspondence[12, 13, 14]. Spiritually the present basic idea overlaps with this approach. The characteristic points of this paper are: a) We do *not* rely on the 5D supergravity; b) We do *not* quantize the gravitational(metric) field; c) The divergence problem is solved by reducing the degree of freedom of the system, where we require, not higher symmetries, but some restriction based on the *minimal area principle*; d) No local counterterms are necessary. @

In the previous paper[15], we investigated the 5D electromagnetism in the *flat* geometry. The results show the *renormalization of the compactification size l* .

$$E_{Cas}^W/\Lambda l = -\frac{\alpha}{l^4} (1 - 4c \ln(l\Lambda)) = -\frac{\alpha}{l'^4} \quad , \quad \beta = \frac{\partial}{\partial(\ln \Lambda)} \ln \frac{l'}{l} = c \quad , \quad (3)$$

where α and c are some numbers. They are, at present, not fixed, but are numerically obtained depending on the weight function W . The aim of this paper is to examine how the above results change for the 5D warped geometry case. One additional massive parameter, that is, the warp (bulk curvature) parameter ω appears. This introduction of the "thickness" $1/\omega$ comes from the expectation that it softens the UV-singularity, which is the same situation as in the string theory.

2. Kaluza-Klein expansion approach

In order to analyze the 5D EM-theory, we start with 5D *massive* vector theory.

$$S_{5dV} = \int d^4x dz \sqrt{-G} \left(-\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} m^2 A^M A_M \right) , \quad F_{MN} = \partial_M A_N - \partial_N A_M , \\ ds^2 = \frac{1}{\omega^2 z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) = G_{MN} dX^M dX^N \quad , \quad G \equiv \det G_{AB} \quad . \quad (4)$$

The 5D vector mass, m , is regarded as a IR-regularization parameter. In the limit, $m = 0$, the above one has the 5D local-gauge symmetry. Casimir energy is given by some integral where the (modified) Bessel functions, with the index $\nu = \sqrt{1 + \frac{m^2}{\omega^2}}$, appear. Hence the 5D EM limit is given by $\nu = 1$ ($m = 0$). We consider, however, the *imaginary* mass case $m = i\omega$ ($m^2 = -\omega^2$, $\nu = 0$) mainly for the simplicity. We can simplify the model furthermore. Instead of analyzing the $m^2 = -\omega^2$ of the massive vector (4), we take the 5D massive *scalar* theory on AdS_5 with $m^2 = -4\omega^2$, $\nu = \sqrt{4 + m^2/\omega^2} = 0$.

$$\mathcal{L} = \sqrt{-G} \left(-\frac{1}{2} \nabla^A \Phi \nabla_A \Phi - \frac{1}{2} m^2 \Phi^2 \right) , \quad ds^2 = G_{AB} dX^A dX^B , \quad \nabla^A \nabla_A \Phi - m^2 \Phi + J = 0 , \quad (5)$$

² The gauge independence was confirmed in Ref.[5].

where $\Phi(X) = \Phi(x^a, z)$ is the 5D scalar field. The integral region is given by

$$-\frac{1}{T} \leq z \leq -\frac{1}{\omega} \quad \text{or} \quad \frac{1}{\omega} \leq z \leq \frac{1}{T} \quad (-l \leq y \leq l, |z| = \frac{1}{\omega} e^{\omega|y|}) \quad , \quad \frac{1}{T} \equiv \frac{1}{\omega} e^{\omega l} \quad , \quad (6)$$

where we take into account Z_2 symmetry: $z \leftrightarrow -z$. ω is the bulk curvature (AdS₅ parameter) and T^{-1} is the size of the extra space (Infrared parameter). In this section, we present the standard expressions, that is, those obtained by the Kaluza-Klein expansion. The Casimir energy E_{Cas} is given by

$$\begin{aligned} e^{-T^{-4}E_{Cas}} &= \int \mathcal{D}\Phi \exp\{i \int d^5X \mathcal{L}\} \\ &= \int \mathcal{D}\Phi_p(z) \exp \left[i \int \frac{d^4p}{(2\pi)^4} 2 \int_{1/\omega}^{1/T} dz \left\{ \frac{1}{2} \Phi_p(z) s(z) (s(z)^{-1} \hat{L}_z - p^2) \Phi_p(z) \right\} \right] \\ &= \int \prod_n dc_n(p) \exp \left[\int \frac{d^4p_E}{(2\pi)^4} \sum_n \left\{ -\frac{1}{2} c_n(p)^2 (p_E^2 + M_n^2) \right\} \right] = \exp \sum_{n,p} \left\{ -\frac{1}{2} \ln(p_E^2 + M_n^2) \right\} \quad , \quad (7) \end{aligned}$$

where $\Phi_p(z)$ is the partially(4D world only)-Fourier-transformed one of $\Phi(X)$. $\Phi_p(z)$ is expressed in the expansion form using the eigen functions, $\psi_n(z)$, of this AdS₅ system.

$$\begin{aligned} \Phi_p(z) &= \sum_n c_n(p) \psi_n(z), \{s(z)^{-1} \hat{L}_z + M_n^2\} \psi_n(z) = 0, \hat{L}_z \equiv \frac{d}{dz} \frac{1}{(\omega z)^3} \frac{d}{dz} - \frac{m^2}{(\omega z)^5}, \\ \psi_n(z) &= -\psi_n(-z) \quad \text{for} \quad P = - \quad ; \quad \psi_n(z) = \psi_n(-z) \quad \text{for} \quad P = + \quad . \quad (8) \end{aligned}$$

where $s(z) = \frac{1}{(\omega z)^3}$. The expression (7) is the familiar one of the Casimir energy.

3. Heat-Kernel Approach and Position/Momentum Propagator

Eq.(7) is the expression of E_{Cas} by the KK-expansion. In this section, the same quantity is re-expressed in a *closed* form using the heat-kernel method and the P/M propagator. First we can express it, using the heat equation solution, as follows.

$$\begin{aligned} e^{-T^{-4}E_{Cas}} &= (\text{const}) \times \exp \left[T^{-4} \int \frac{d^4p}{(2\pi)^4} 2 \int_0^\infty \frac{1}{2} \frac{dt}{t} \text{Tr} H_p(z, z'; t) \right] \quad , \\ \text{Tr} H_p(z, z'; t) &= \int_{1/\omega}^{1/T} s(z) H_p(z, z; t) dz \quad , \quad \left\{ \frac{\partial}{\partial t} - (s^{-1} \hat{L}_z - p^2) \right\} H_p(z, z'; t) = 0 \quad . \quad (9) \end{aligned}$$

The heat kernel $H_p(z, z'; t)$ is formally solved, using the Dirac's bra and ket vectors $\langle z|, |z\rangle$, as

$$H_p(z, z'; t) = \langle z| e^{-(s^{-1} \hat{L}_z + p^2)t} |z'\rangle \quad . \quad (10)$$

We here introduce the position/momentum propagators G_p^\mp as follows.

$$G_p^\mp(z, z') \equiv \int_0^\infty dt H_p(z, z'; t) = \sum_{n \in \mathbf{Z}} \frac{1}{M_n^2 + p^2} \frac{1}{2} \{ \psi_n(z) \psi_n(z') \mp \psi_n(z) \psi_n(-z') \} \quad . \quad (11)$$

They satisfy the following differential equations of *propagators*.

$$(\hat{L}_z - p^2 s(z)) G_p^\mp(z, z') = \sum_{n \in \mathbf{Z}} \frac{\{ \psi_n(z) \psi_n(z') \mp \psi_n(z) \psi_n(-z') \}}{2} = \begin{cases} \epsilon(z) \epsilon(z') \hat{\delta}(|z| - |z'|) & P = -1 \\ \hat{\delta}(|z| - |z'|) & P = 1 \end{cases} \quad (12)$$

Therefore the Casimir energy E_{Cas} is given by

$$\begin{aligned} -E_{Cas}^-(\omega, T) &= \int \frac{d^4 p_E}{(2\pi)^4} 2 \int_0^\infty \frac{dt}{t} 2 \int_{1/\omega}^{1/T} dz s(z) H_{p_E}(z, z; t) \\ &= \int \frac{d^4 p_E}{(2\pi)^4} 2 \int_0^\infty \frac{dt}{t} 2 \int_{1/\omega}^{1/T} dz s(z) \left\{ \sum_{n \in \mathbf{Z}} e^{-(M_n^2 + p_E^2)t} \psi_n(z)^2 \right\} , \end{aligned} \quad (13)$$

where $s(z) = 1/(\omega z)^3$. The momentum symbol p_E indicates Euclideanization. This expression leads to the same treatment as the previous section. Note that the above expression shows the *negative definiteness* of E_{Cas}^- . Finally we obtain the following useful expression of the Casimir energy for $P = \mp$.

$$-E_{Cas}^\mp(\omega, T) = \int \frac{d^4 p_E}{(2\pi)^4} \int_{1/\omega}^{1/T} dz s(z) \int_{p_E^2}^\infty \{G_k^\mp(z, z)\} dk^2 . \quad (14)$$

The P/M propagators G_p^\mp in (11) and (14) can be expressed in a *closed* form. Taking the *Dirichlet* condition at all fixed points, the expression for the fundamental region ($1/\omega \leq z \leq z' \leq 1/T$) is given by

$$G_p^\mp(z, z') = \mp \frac{\omega^3}{2} z^2 z'^2 \frac{\{\mathbf{I}_0(\frac{\tilde{p}}{\omega})\mathbf{K}_0(\tilde{p}z) \mp \mathbf{K}_0(\frac{\tilde{p}}{\omega})\mathbf{I}_0(\tilde{p}z)\} \{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\tilde{p}z') \mp \mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\tilde{p}z')\}}{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\frac{\tilde{p}}{\omega}) - \mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\frac{\tilde{p}}{\omega})}, \quad (15)$$

where $\tilde{p} \equiv \sqrt{p^2}, p^2 \geq 0$. We can express Casimir energy in terms of the following functions $F^\mp(\tilde{p}, z)$.

$$\begin{aligned} -E_{Cas}^{\Lambda, \mp}(\omega, T) &= \int \frac{d^4 p_E}{(2\pi)^4} \bigg|_{\tilde{p} \leq \Lambda} \int_{1/\omega}^{1/T} dz F^\mp(\tilde{p}, z) , \\ F^\mp(\tilde{p}, z) &\equiv s(z) \int_{p_E^2}^{\Lambda^2} \{G_k^\mp(z, z)\} dk^2 = \frac{2}{(\omega z)^3} \int_{\tilde{p}}^\Lambda \tilde{k} G_k^\mp(z, z) d\tilde{k} \equiv \int_{\tilde{p}}^\Lambda \mathcal{F}^\mp(\tilde{k}, z) d\tilde{k} , \end{aligned} \quad (16)$$

where $\mathcal{F}^\mp(\tilde{k}, z)$ are the integrands of $F^\mp(\tilde{p}, z)$ and $\tilde{p} = \sqrt{p_E^2}$. Here we introduce the UV cut-off parameter Λ for the 4D momentum space. In Fig.2, we show the behavior of $\mathcal{F}^-(\tilde{k}, z)$. The table-shape graph says the "Rayleigh-Jeans" dominance. That is, for the wide-range region (\tilde{p}, z) satisfying both $\tilde{p}(z - \frac{1}{\omega}) \gg 1$ and $\tilde{p}(\frac{1}{T} - z) \gg 1$,

$$\mathcal{F}^-(\tilde{p}, z) \approx \frac{1}{2} , \quad \mathcal{F}^+(\tilde{p}, z) \approx \frac{1}{2} , \quad (\tilde{p}, z) \in \{(\tilde{p}, z) | \tilde{p}(z - \frac{1}{\omega}) \gg 1 \text{ and } \tilde{p}(\frac{1}{T} - z) \gg 1\} . \quad (17)$$

4. UV and IR Regularization Parameters and Evaluation of Casimir Energy

The integral region of the above equation (16) is displayed in Fig.3. In the figure, we introduce the regularization cut-offs for the 4D-momentum integral, $\mu \leq \tilde{p} \leq \Lambda$. As for the extra-coordinate integral, it is the finite interval, $1/\omega \leq z \leq 1/T = e^{\omega l}/\omega$, hence we need not introduce further regularization parameters. For simplicity, we take the following IR cutoff of 4D momentum : $\mu = \Lambda \cdot \frac{T}{\omega} = \Lambda e^{-\omega l}$. Hence the new regularization parameter is Λ only.

Let us evaluate the (Λ, T) -regularized value of (16).

$$-E_{Cas}^{\Lambda, \mp}(\omega, T) = \frac{2\pi^2}{(2\pi)^4} \int_\mu^\Lambda d\tilde{p} \int_{1/\omega}^{1/T} dz \tilde{p}^3 F^\mp(\tilde{p}, z) , \quad F^\mp(\tilde{p}, z) = \frac{2}{(\omega z)^3} \int_{\tilde{p}}^\Lambda \tilde{k} G_k^\mp(z, z) d\tilde{k} . \quad (18)$$

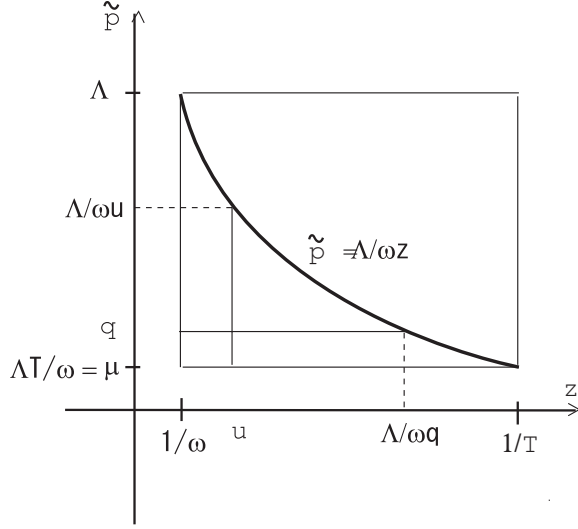


Figure 3. Space of (z, \tilde{p}) for the integration. The hyperbolic curve will be used in Sec.5.

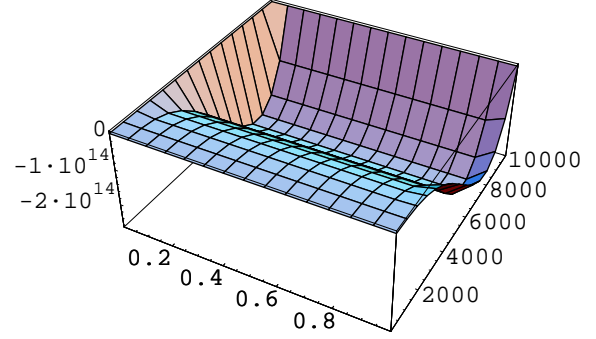


Figure 4. Behaviour of $(-1/2)\tilde{p}^3 F^-(\tilde{p}, z)$ (18). $T = 1, \omega = 10^4, \Lambda = 10^4$. $1.0001/\omega \leq z < 0.9999/T, \Lambda T/\omega \leq \tilde{p} \leq \Lambda$.

The integral region of (\tilde{p}, z) is the *rectangle* shown in Fig.3. Note that eq.(18) is the *rigorous* expression of the (Λ, T) -regularized Casimir energy. We show the behavior of $(-1/2)\tilde{p}^3 F^-(\tilde{p}, z)$ taking the values $\omega = 10^4, T = 1$ in Fig.4 ($\Lambda = 10^4$).³ Behavior along \tilde{p} -axis does not so much depend on z . A valley runs parallel to the z -axis with the bottom line at the fixed ratio of $\tilde{p}/\Lambda \sim 0.75$. The depth of the valley is proportional to Λ^4 . Because E_{Cas} is the (\tilde{p}, z) 'flat-plane' integral of $\tilde{p}^3 F(\tilde{p}, z)$, the *volume* inside the valley is the quantity E_{Cas} . Hence it is easy to see E_{Cas} is proportional to Λ^5 . This is the same situation as the flat case. Importantly, (18) shows the *scaling* behavior for large values of Λ and $1/T$. From a *close* numerical analysis of (\tilde{p}, z) -integral (18), we have confirmed : Eq.(4A) $E_{Cas}^{\Lambda, -}(\omega, T) = \frac{2\pi^2}{(2\pi)^4} \times \left[-0.0250 \frac{\Lambda^5}{T} \right]$. It does *not* depend on ω and has no $\ln \frac{\Lambda}{T}$ -term. (Note: $0.025 = 1/40$.) Compared with the flat case, we see the factor T^{-1} plays the role of *IR parameter* of the extra space. We note that the behavior of Fig.4 is similar to the Rayleigh-Jeans's region (small momentum region) of the Planck's radiation formula (Fig.1) in the sense that $\tilde{p}^3 F(\tilde{p}, z) \propto \tilde{p}^3$ for $\tilde{p} \ll \Lambda$.

Finally we notice, from the Fig.4, the approximate form of $F(\tilde{p}, z)$ for the large Λ and $1/T$ is given by : Eq.(4B) $F^\mp(\tilde{p}, z) \approx \frac{f}{2} \Lambda (1 - \frac{\tilde{p}}{\Lambda})$, $f = 1$. It does *not* depend on z, ω and T . f is the degree of freedom. The above result is consistent with (17).

5. UV and IR Regularization Surfaces, Principle of Minimal Area and Renormalization Flow

The advantage of the new approach is that the KK-expansion is replaced by the integral of the extra dimensional coordinate z and all expressions are written in the *closed* (not expanded) form. The Λ^5 -divergence, (4A), shows the notorious problem of the higher dimensional theories, as in the flat case. In spite of all efforts of the past literature, we have not succeeded in defining the higher-dimensional theories. (The divergence causes problems. The famous example is the divergent *cosmological constant* in the gravity-involving theories. [4]) Here we notice that the divergence problem can be solved if we find a way to *legitimately restrict the integral region in (\tilde{p}, z) -space*.

³ The requirement for the three parameters ω, T, Λ is $\Lambda \gg \omega \gg T$. See ref.[18] for the discussion about the hierarchy Λ, ω, T .

One proposal of this was presented by Randall and Schwartz[16]. They introduced the *position-dependent cut-off*, $\mu < \tilde{p} < \Lambda/\omega u$, $u \in [1/\omega, 1/T]$, for the 4D-momentum integral in the "brane" located at $z = u$. See Fig.3. The total integral region is the lower part of the *hyperbolic curve* $\tilde{p} = \Lambda/\omega z$. They succeeded in obtaining the *finite* β -function of the 5D warped vector model. We have confirmed that the value E_{Cas} of (18), when the Randall-Schwartz integral region (Fig.3) is taken, is proportional to Λ^5 . The close numerical analysis says

$$E_{Cas}^{-RS}(\omega, T) = \frac{2\pi^2}{(2\pi)^4} \int_{\mu}^{\Lambda} dq \int_{1/\omega}^{\Lambda/\omega q} dz q^3 F^-(q, z) = \frac{2\pi^2}{(2\pi)^4} \int_{1/\omega}^{1/T} du \int_{\mu}^{\Lambda/\omega u} d\tilde{p} \tilde{p}^3 F^-(\tilde{p}, u) \\ = \frac{2\pi^2}{(2\pi)^4} \frac{\Lambda^5}{\omega} \left\{ -1.58 \times 10^{-2} - 1.69 \times 10^{-4} \ln \frac{\Lambda}{\omega} \right\}, \quad (19)$$

which is *independent* of T . This shows the divergence situation does *not* improve compared with the non-restricted case of (4A). T of (4A) is replaced by the warp parameter ω . This is contrasting with the flat case where $E_{Cas}^{RS} \propto -\Lambda^4$. The UV-behavior, however, *does improve* if we can choose the parameter Λ in the way: $\Lambda \propto \omega$. This fact shows the parameter ω "smoothes" the UV-singularity to some extent.

Although they claim the holography is behind the procedure, the legitimacy of the restriction looks less obvious. We have proposed an alternate approach and given a legitimate explanation within the 5D QFT[17, 19, 15, 20]. Here we closely examine the *new regularization*. On the "3-brane" at $z = 1/\omega$, we introduce the IR-cut-off $\mu = \Lambda \cdot \frac{T}{\omega}$ and the UV-cut-off Λ

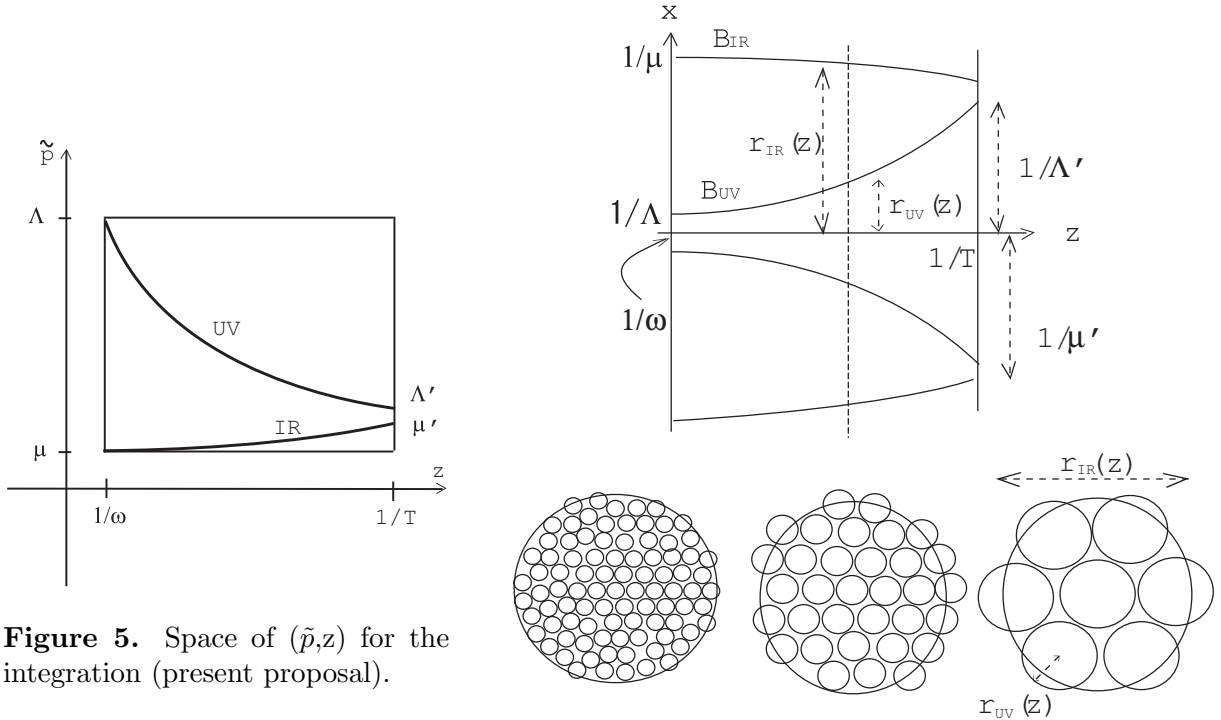


Figure 5. Space of (\tilde{p}, z) for the integration (present proposal).

Figure 6. Regularization Surface B_{IR} and B_{UV} in the 5D coordinate space (x^μ, z) .

($\mu \ll \Lambda$) in the way : Eq.(5A) $\mu \ll \Lambda$ ($T \ll \omega$). See Fig.5. This is legitimate in the sense that we generally do this procedure in the 4D *renormalizable* theories. (Here we are considering those 5D theories that are *renormalizable* in "3-branes". Examples are 5D free theories (present

model), 5D electromagnetism[15], 5D Φ^4 -theory, 5D Yang-Mills theory, e.t.c..) In the same reason, on the "3-brane" at $z = 1/T$, we may have another set of IR and UV-cutoffs, μ' and Λ' . We consider the case: Eq.(5B) $\mu' \leq \Lambda'$, $\Lambda' \ll \Lambda$, $\mu \sim \mu'$. This case will lead us to introduce the *renormalization flow*. (See the later discussion.) We claim here, as for the regularization treatment of the "3-brane" located at other points z ($1/\omega < z \leq 1/T$), the regularization parameters are determined by the *minimal area principle*. To explain it, we move to the 5D coordinate space (x^μ, z) . See Fig.6. The \tilde{p} -expression can be replaced by $\sqrt{x_\mu x^\mu}$ -expression by the *reciprocal relation* : Eq.(5C) $\sqrt{x_\mu(z)x^\mu(z)} \equiv r(z) \leftrightarrow \frac{1}{\tilde{p}(z)}$. The UV and IR cutoffs change their values along z -axis and their trajectories make *surfaces* in the 5D bulk space (x^μ, z) . We *require* the two surfaces do *not cross* for the purpose of the renormalization group interpretation (discussed later). We call them UV and IR regularization (or boundary) surfaces(B_{UV}, B_{IR}). The cross sections of the regularization surfaces at z are the spheres S^3 with the radii $r_{UV}(z)$ and $r_{IR}(z)$. Here we consider the Euclidean space for simplicity. The UV-surface is stereographically shown in Fig.7 and reminds us of the *closed string* propagation. Note that the boundary surface B_{UV} (and B_{IR}) is the 4 dimensional manifold.

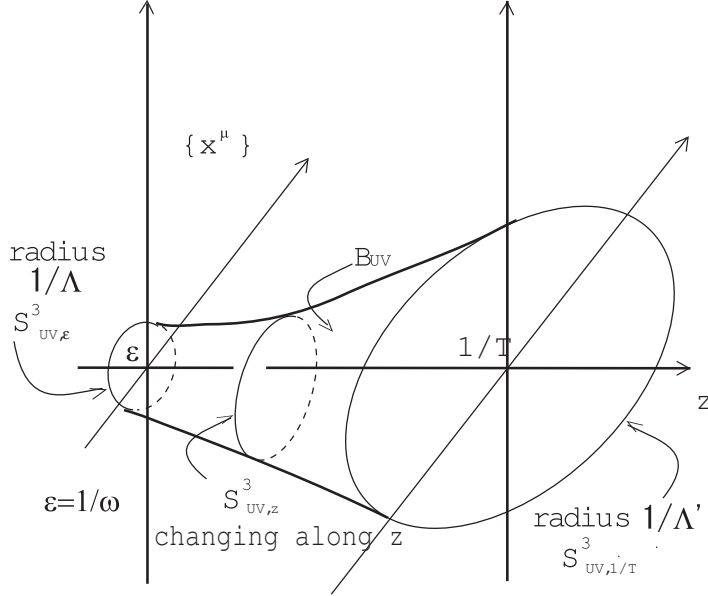


Figure 7. UV regularization surface (B_{UV}) in 5D coordinate space.

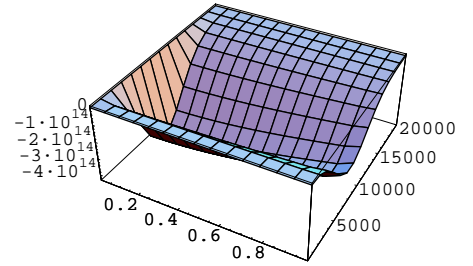


Figure 8. Behavior of $(-N_1/2)\tilde{p}^3 W_1(\tilde{p}, z) F^-(\tilde{p}, z)$ (elliptic suppression). $\Lambda = 20000$, $\omega = 5000$, $T = 1$. $1.0001/\omega \leq z \leq 0.9999/T$, $\mu = \Lambda T/\omega \leq \tilde{p} \leq \Lambda$.

The 5D volume region bounded by B_{UV} and B_{IR} is the integral region of the Casimir energy E_{Cas} . The forms of $r_{UV}(z)$ and $r_{IR}(z)$ can be determined by the *minimal area principle*.

$$3 + \frac{4}{z} r' r - \frac{r'' r}{r'^2 + 1} = 0 \quad , \quad r' \equiv \frac{dr}{dz} \quad , \quad r'' \equiv \frac{d^2 r}{dz^2} \quad , \quad 1/\omega \leq z \leq 1/T \quad . \quad (20)$$

We have confirmed, by numerically solving the above differential equation (Runge-Kutta), those curves that show the flow of renormalization really occur. The results imply the *boundary conditions* determine the property of the renormalization flow.

The present regularization scheme gives the *renormalization group* interpretation to the change of physical quantities along the extra axis. See Fig.6. ⁴ In the "3-brane" located

⁴ This part is contrasting with AdS/CFT approach where the renormalization flow comes from the Einstein equation of 5D supergravity.

at z , the UV-cutoff is $r_{UV}(z)$ and the regularization surface is the sphere S^3 with the radius $r_{UV}(z)$. The IR-cutoff is $r_{IR}(z)$ and the regularization surface is the another sphere S^3 with the radius $r_{IR}(z)$. We can regard the regularization integral region as the *sphere lattice* of the following properties: a) A unit lattice (cell) is the sphere S^3 with radius $r_{UV}(z)$ and its inside; b) Total lattice is the sphere S^3 with radius $r_{IR}(z)$ and its inside; c) The integration region of this regularization is made of many cells and the total number of cells is $\text{const.} \times \left(\frac{r_{IR}(z)}{r_{UV}(z)}\right)^4$. The total number of cells changes from $(\frac{\Lambda}{\mu})^4$ at $z = 1/\omega$ to $(\frac{\Lambda'}{\mu'})^4$ at $z = 1/T$. Along the z -axis, the number increases or decreases as : Eq.(5D) $\left(\frac{r_{IR}(z)}{r_{UV}(z)}\right)^4 \equiv N(z)$. For the "scale" change $z \rightarrow z + \Delta z$, N changes as : Eq.(5E) $\Delta(\ln N) = 4 \frac{\partial}{\partial z} \{\ln(\frac{r_{IR}(z)}{r_{UV}(z)})\} \cdot \Delta z$. When the system has some coupling $g(z)$, the renormalization group $\tilde{\beta}(g)$ -function (along the extra axis) is expressed as

$$\tilde{\beta} = \frac{\Delta(\ln g)}{\Delta(\ln N)} = \frac{1}{\Delta(\ln N)} \frac{\Delta g}{g} = \frac{1}{4} \frac{1}{\frac{\partial}{\partial z} \ln(\frac{r_{IR}(z)}{r_{UV}(z)})} \frac{1}{g} \frac{\partial g}{\partial z} \quad , \quad (21)$$

where $g(z)$ is a renormalized coupling at z .⁵

We have explained, in this section, that the *minimal area principle* determines the flow of the regularization surfaces.

6. Weight Function and Casimir Energy Evaluation

In the expression (14), the Casimir energy is written by the integral in the (\tilde{p}, z) -space over the range: $1/\omega \leq z \leq 1/T$, $0 \leq \tilde{p} \leq \infty$. In Sec.5, we have seen *the integral region should be properly restricted* because the cut-off region in the 4D world *changes along the extra-axis* obeying the bulk (warped) geometry (*minimal area principle*). We can expect the singular behavior (UV divergences) reduces by the integral-region restriction, but the concrete evaluation along the proposed prescription is practically not easy. In this section, we consider an alternate approach which respects the *minimal area principle* and evaluate the Casimir energy.

We introduce, instead of restricting the integral region, a *weight function* $W(\tilde{p}, z)$ in the (\tilde{p}, z) -space for the purpose of suppressing UV and IR divergences of the Casimir Energy.

$$-E_{Cas}^{\mp W}(\omega, T) \equiv \int \frac{d^4 p_E}{(2\pi)^4} \int_{1/\omega}^{1/T} dz W(\tilde{p}, z) F^{\mp}(\tilde{p}, z) \quad , \quad \tilde{p} = \sqrt{p_4^2 + p_1^2 + p_2^2 + p_3^2} \quad ,$$

Examples of $W(\tilde{p}, z)$: $W(\tilde{p}, z) =$

$$\left\{ \begin{array}{ll} (N_1)^{-1} e^{-(1/2)\tilde{p}^2/\omega^2 - (1/2)z^2 T^2} \equiv W_1(\tilde{p}, z), \quad N_1 = 1.711/8\pi^2 & \text{elliptic suppr.} \\ (N_2)^{-1} e^{-\tilde{p}zT/\omega} \equiv W_2(\tilde{p}, z), \quad N_2 = 2\frac{\omega^3}{T^3}/8\pi^2 & \text{hyperbolic suppr.1} \\ (N_8)^{-1} e^{-1/2(\tilde{p}^2/\omega^2 + 1/z^2 T^2)} \equiv W_8(\tilde{p}, z), \quad N_8 = 0.4177/8\pi^2 & \text{reciprocal suppr.1} \end{array} \right. \quad (22)$$

where $F^{\mp}(\tilde{p}, z)$ are defined in (16). In the above, we list some examples expected for the weight function $W(\tilde{p}, z)$. W_2 is regarded to correspond to the regularization taken by Randall-Schwartz. How to specify the form of W is the subject of the next section. We show the shape of the energy integrand $(-1/2)\tilde{p}^3 W_1(\tilde{p}, z) F^-(\tilde{p}, z)$ in Fig.8. We notice the valley-bottom line $\tilde{p} \approx 0.75\Lambda$, which appeared in the un-weighted case (Fig.4), is replaced by a new line: $\tilde{p}^2 + z^2 \times \omega^2 T^2 \approx \text{const.}$ It is located *away from* the original Λ -effected line ($\tilde{p} \sim 0.75\Lambda$).

⁵ Here we consider an interacting theory, such as 5D Yang-Mills theory and 5D Φ^4 theory, where the coupling $g(z)$ is the renormalized one in the '3-brane' at z .

We can check the divergence (scaling) behavior of $E_{Cas}^{\mp W}$ by *numerically* evaluating the (\tilde{p}, z) -integral (22) for the rectangle region of Fig.3.

$$-E_{Cas}^W = \begin{cases} \frac{\omega^4}{T} \Lambda \times 1.2 \left\{ 1 + 0.11 \ln \frac{\Lambda}{\omega} - 0.10 \ln \frac{\Lambda}{T} \right\} & \text{for } W_1 \\ \frac{T^2}{\omega^2} \Lambda^4 \times 0.062 \left\{ 1 + 0.03 \ln \frac{\Lambda}{\omega} - 0.08 \ln \frac{\Lambda}{T} \right\} & \text{for } W_2 \\ \frac{\omega^4}{T} \Lambda \times 1.6 \left\{ 1 + 0.09 \ln \frac{\Lambda}{\omega} - 0.10 \ln \frac{\Lambda}{T} \right\} & \text{for } W_8 \end{cases} \quad (23)$$

The suppression behavior of W_2 improves, compared with (19) by Randall-Schwartz. The quintic divergence of (19) reduces to the quartic divergence in the present approach of W_2 . The hyperbolic suppressions, however, are still insufficient for the renormalizability. After dividing by the normalization factor, ΛT^{-1} , the cubic divergence remains. The desired cases are others. The Casimir energy for each case consists of three terms. The first terms give *finite* values after dividing by the overall normalization factor ΛT^{-1} . The last two terms are proportional to $\log \Lambda$ and show the *anomalous scaling*. Their contributions are order of 10^{-1} to the first leading terms. The second ones ($\ln \frac{\Lambda}{\omega}$) contribute positively while the third ones ($\ln \frac{\Lambda}{T}$) negatively. They give, after normalizing the factor Λ/T , *only* the *log-divergence*.

$$E_{Cas}^W / \Lambda T^{-1} = -\alpha \omega^4 (1 - 4c \ln(\Lambda/\omega) - 4c' \ln(\Lambda/T)) \quad , \quad (24)$$

where α, c and c' can be read from (23) depending on the choice of W . This means the 5D Casimir energy is *finitely* obtained by the ordinary renormalization of the warp factor ω . (See the final section.) In the above result of the warped case, the IR parameter l in the flat result (3) is replaced by the inverse of the warp factor ω .

So far as the legitimate reason of the introduction of $W(\tilde{p}, y)$ is not clear, we should regard this procedure as a *regularization* to define the higher dimensional theories. We give a clear definition of $W(\tilde{p}, y)$ and a legitimate explanation in the next section. It should be done, in principle, in a consistent way with the bulk geometry and the gauge principle.

7. Meaning of Weight Function and Quantum Fluctuation of Coordinates and Momenta

In the previous work[15], we have presented the following idea to define the weight function $W(\tilde{p}, z)$. In the evaluation (22):

$$-E_{Cas}^W(\omega, T) = \int \frac{d^4 p_E}{(2\pi)^4} \int_{1/\omega}^{1/T} dz W(\tilde{p}, z) F^{\mp}(\tilde{p}, z) = \frac{2\pi^2}{(2\pi)^4} \int d\tilde{p} \int_{1/\omega}^{1/T} dz \tilde{p}^3 W(\tilde{p}, z) F^{\mp}(\tilde{p}, z) \quad , \quad (25)$$

the (\tilde{p}, z) -integral is over the rectangle region shown in Fig.5 (with $\Lambda \rightarrow \infty$ and $\mu \rightarrow 0$). $F^{\mp}(\tilde{p}, z)$ is explicitly given in (16). Following Feynman[21], we can replace the integral by the summation over all possible pathes $\tilde{p}(z)$.

$$-E_{Cas}^W(\omega, T) = \int \mathcal{D}\tilde{p}(z) \int_{1/\omega}^{1/T} dz S[\tilde{p}(z), z] \quad , \quad S[\tilde{p}(z), z] = \frac{2\pi^2}{(2\pi)^4} \tilde{p}(z)^3 W(\tilde{p}(z), z) F^{\mp}(\tilde{p}(z), z). \quad (26)$$

There exists the *dominant path* $\tilde{p}_W(z)$ which is determined by the minimal principle : $\delta S = 0$.

$$\text{Dominant Path } \tilde{p}_W(z) : \quad \frac{d\tilde{p}}{dz} = \frac{-\frac{\partial \ln(WF)}{\partial z}}{\frac{3}{\tilde{p}} + \frac{\partial \ln(WF)}{\partial \tilde{p}}} \quad . \quad (27)$$

Hence it is fixed by $W(\tilde{p}, z)$. An example is the valley-bottom line in Fig.8. On the other hand, there exists another independent path: the minimal surface curve $r_g(z)$.

$$\text{Minimal Surface Curve } r_g(z) : \quad 3 + \frac{4}{z} r' r - \frac{r'' r}{r'^2 + 1} = 0 \quad , \quad \frac{1}{\omega} \leq z \leq \frac{1}{T} \quad , \quad (28)$$

which is obtained by the *minimal area principle*: $\delta A = 0$ where

$$ds^2 = (\delta_{ab} + \frac{x^a x^b}{(r r')^2}) \frac{dx^a dx^b}{\omega^2 z^2} \equiv g_{ab}(x) dx^a dx^b, A = \int \sqrt{\det g_{ab}} d^4 x = \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \sqrt{r'^2 + 1} r^3 dz. \quad (29)$$

Hence $r_g(z)$ is fixed by the *induced geometry* $g_{ab}(x)$. Here we put the *requirement*[15]: Eq.(7A) $\tilde{p}_W(z) = \tilde{p}_g(z)$, where $\tilde{p}_g \equiv 1/r_g$. This means the following things. We *require* the dominant path coincides with the minimal surface line $\tilde{p}_g(z) = 1/r_g(z)$ which is defined independently of $W(\tilde{p}, z)$. In other words, $W(\tilde{p}, z)$ is defined here by the induced geometry $g_{ab}(x)$. In this way, we can connect the integral-measure over the 5D-space with the (bulk) geometry. We have confirmed the (approximate) coincidence by the numerical method.

In order to most naturally accomplish the above requirement, we can go to a *new step*. Namely, we *propose* to *replace* the 5D space integral with the weight W , (25), by the following *path-integral*. We *newly define* the Casimir energy in the higher-dimensional theory as follows.

$$- \mathcal{E}_{Cas}(\omega, T, \Lambda) \equiv \int_{1/\Lambda}^{1/\mu} d\rho \int_{r(1/\omega)}^{r(1/T)} \prod_{a,z} \mathcal{D}x^a(z) F(\frac{1}{r}, z) \exp \left[-\frac{1}{2\alpha'} \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \sqrt{r'^2 + 1} r^3 dz \right], \quad (30)$$

$= \rho$

where $\mu = \Lambda T/\omega$ and the limit $\Lambda T^{-1} \rightarrow \infty$ is taken. The string (surface) tension parameter $1/2\alpha'$ is introduced. (Note: Dimension of α' is [Length]⁴.) The square-bracket ([\dots])-parts of (30) are $-\frac{1}{2\alpha'} \text{Area} = -\frac{1}{2\alpha'} \int \sqrt{\det g_{ab}} d^4 x$ where g_{ab} is the induced metric on the 4D surface. $F(\tilde{p}, z)$ is defined in (22) or (16) and shows the *field-quantization* of the bulk scalar (EM) fields. In the above expression, we have followed the path-integral formulation of the *density matrix* (See Feynman's text[21]). The validity of the above definition is based on the following points: a) When the weight part ($\exp [\dots]$ -part) is 1, the proposed quantity \mathcal{E}_{Cas} is equal to E_{Cas}^W , (25), with $W = 1$; b) The leading path is given by $r_g(z) = 1/p_g(z)$, (28); c) The proposed definition, (30), clearly shows the 4D space-coordinates x^a or the 4D momentum-coordinates p^a are *quantized* (quantum-statistically, not field-theoretically) with the Euclidean time z and the "area Hamiltonian" $A = \int \sqrt{\det g_{ab}} d^4 x$. Note that $F(\tilde{p}, z)$ or $F(1/r, z)$ appears, in (30), as the *energy density operator* in the quantum statistical system of $\{p^a(z)\}$ or $\{x^a(z)\}$.

In the view of the previous paragraph, the treatment of Sec.6 is an *effective action* approach using the (trial) weight function $W(\tilde{p}, z)$. Note that the integral over (p^μ, z) -space, appearing in (16), is the summation over all degrees of freedom of the 5D space(-time) points using the "naive" measure $d^4 p dz$. An important point is that we have the possibility to take another measure for the summation in the case of the higher dimensional QFT. We have adopted, in Sec.6, the new measure $W(p^\mu, z) d^4 p dz$ in such a way that the Casimir energy *does not show physical divergences*. We expect the *direct* evaluation of (30), numerically or analytically, leads to the similar result.

8. Discussion and Conclusion

The log-divergence in (24) is the familiar one in the ordinary QFT. It can be *renormalized* in the following way.

$$\frac{E_{Cas}^W}{\Lambda T^{-1}} = -\alpha \omega^4 \left(1 - 4c \ln\left(\frac{\Lambda}{\omega}\right) - 4c' \ln\left(\frac{\Lambda}{T}\right) \right) = -\alpha (\omega_r)^4, \omega_r = \omega \sqrt[4]{1 - 4c \ln\left(\frac{\Lambda}{\omega}\right) - 4c' \ln\left(\frac{\Lambda}{T}\right)}, \quad (31)$$

where ω_r is the *renormalized* warp factor and ω is the *bare* one. No local counterterms are necessary. Note that this renormalization relation is *exact* (not a perturbative result). In the familiar case of the 4D renormalizable theories, the coefficients c and c' depend on the coupling,

but, in the present case, they are *pure numbers*. It reflects the *interaction between (EM) fields and the boundaries*. When c and c' are sufficiently small we find the renormalization group function for the warp factor ω as

$$|c| \ll 1, |c'| \ll 1, \quad \omega_r = \omega(1 - c \ln(\Lambda/\omega) - c' \ln(\Lambda/T)), \quad \beta \equiv \frac{\partial}{\partial(\ln \Lambda)} \ln \frac{\omega_r}{\omega} = -c - c'. \quad (32)$$

We should notice that, in the flat geometry case, the IR parameter (extra-space size) l is renormalized. In the present warped case, however, the corresponding parameter T is *not renormalized*, but the *warp parameter ω is renormalized*. Depending on the sign of $c + c'$, the 5D bulk curvature ω *flows* as follows. When $c + c' > 0$, the bulk curvature ω decreases (increases) as the measurement energy scale Λ increases (decreases). When $c + c' < 0$, the flow goes in the opposite way. When $c + c' = 0$, ω does not flow ($\beta = 0$) and is given by $\omega_r = \omega(1 + c \ln(\omega/T))$.

The final result (31) is the *new* type Casimir energy, $-\omega^4$. ω appears as a boundary parameter like T . The familiar one is $-T^4$ in the present context. In ref.[22], another type $T^2\omega^2$ was predicted using a "quasi" Warped model (bulk-boundary theory).

Through the Casimir energy calculation, in the higher dimension, we find a way to quantize the higher dimensional theories within the QFT framework. The quantization *with respect to the fields* (except the gravitational fields $G_{AB}(X)$) is done in the standard way. After this step, the expression has the summation *over the 5D space(-time) coordinates or momenta* $\int dz \prod_a dp^a$. We have proposed that this summation should be replaced by the *path-integral* $\int \prod_{a,z} \mathcal{D}p^a(z)$ with the *area* action (Hamiltonian) $A = \int \sqrt{\det g_{ab}} d^4x$ where g_{ab} is the *induced* metric on the 4D surface. This procedure says the 4D momenta p^a (or coordinates x^a) are *quantum statistical* operators and the extra-coordinate z is the inverse temperature (Euclidean time). We recall the similar situation occurs in the standard string approach. The space-time coordinates obey some uncertainty principle[23].

Recently the dark energy (as well as the dark matter) in the universe is a hot subject. It is well-known that the dominant candidate is the cosmological term. We also know the prototype higher-dimensional theory, that is, the 5D KK theory, has predicted so far the *divergent* cosmological constant[4]. This unpleasant situation has been annoying us for a long time. If we apply the present result, the situation drastically improves. The cosmological constant λ appears as: Eq.(8A) $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu} = T_{\mu\nu}^{matter}$, $S = \int d^4x \sqrt{-g} \{ \frac{1}{G_N} (R + \lambda) \} + \int d^4x \sqrt{-g} \{ \mathcal{L}_{matter} \}$, $g = \det g_{\mu\nu}$, where G_N is the Newton's gravitational constant, R is the Riemann scalar curvature. We consider here the 3+1 dim Lorentzian space-time ($\mu, \nu = 0, 1, 2, 3$). The constant λ observationally takes the value : Eq.(8B) $\frac{1}{G_N} \lambda_{obs} \sim \frac{1}{G_N R_{cos}^2} \sim m_\nu^4 \sim (10^{-3} eV)^4$, $\lambda_{obs} \sim \frac{1}{R_{cos}^2} \sim 4 \times 10^{-66} (eV)^2$, where $R_{cos} \sim 5 \times 10^{32} eV^{-1}$ is the cosmological size (Hubble length), m_ν is the neutrino mass.⁶ On the other hand, we have theoretically so far : Eq.(8C) $\frac{1}{G_N} \lambda_{th} \sim \frac{1}{G_N^2} = M_{pl}^4 \sim (10^{28} eV)^4$. This is because the mass scale usually comes from the quantum gravity. (See ref.[26] for the derivation using the Coleman-Weinberg mechanism.) We have the famous huge discrepancy factor : Eq.(8D) $\frac{\lambda_{th}}{\lambda_{obs}} \sim N_{DL}^2$, $N_{DL} \equiv M_{pl} R_{cos} \sim 6 \times 10^{60}$, where N_{DL} is the Dirac's large number[27]. If we use the present result (31), we can obtain a natural choice of T, ω and Λ as follows. By identifying $T^{-4} E_{Cas} = -\alpha_1 \Lambda T^{-1} \omega^4 / T^4$ with $\int d^4x \sqrt{-g} (1/G_N) \lambda_{ob} = R_{cos}^2 (1/G_N)$, we obtain the following relation: Eq.(8E) $N_{DL}^2 = R_{cos}^2 \frac{1}{G_N} = -\alpha_1 \frac{\omega^4 \Lambda}{T^5}$, α_1 : some coefficient. The warped (AdS₅) model predicts the cosmological constant *negative*, hence we have interest only in its absolute value. We take the following choice for Λ and ω : Eq.(8F) $\Lambda = M_{pl} \sim 10^{19} GeV$, $\omega \sim \frac{1}{\sqrt[4]{G_N R_{cos}^2}} = \sqrt{\frac{M_{pl}}{R_{cos}}} \sim m_\nu \sim 10^{-3} eV$. The choice

⁶ The relation $m_\nu \sim \sqrt{M_{pl}/R_{cos}} = \sqrt{1/R_{cos} \sqrt{G_N}}$, which appears in some extra dimension model[24, 25], is used. The neutrino mass is, at least empirically, located at the *geometrical average* of two extreme ends of the mass scales in the universe.

for Λ is accepted in that the largest known energy scale is the Planck energy. The choice for ω comes from the experimental bound for the Newton's gravitational force.

As shown above, we have the standpoint that the cosmological constant is mainly made from the Casimir energy. We do not yet succeed in obtaining the value α_1 negatively, but succeed in obtaining the finiteness of the cosmological constant and its gross absolute value. The smallness of the value is naturally explained by the renormalization flow as follows. Because we already know the warp parameter ω flows (32), the $\lambda_{obs} \sim 1/R_{cos}^2$ expression (8F), $\lambda_{obs} \propto \omega^4$, says that the *smallness of the cosmological constant comes from the renormalization flow* for the non asymptotic-free case ($c + c' < 0$ in (32)).

The IR parameter T , the normalization factor Λ/T in (24) and the IR cutoff $\mu = \Lambda \frac{T}{\omega}$ are given by : Eq.(8G) $T = R_{cos}^{-1}(N_{DL})^{1/5} \sim 10^{-20} eV$, $\frac{\Lambda}{T} = (N_{DL})^{4/5} \sim 10^{50}$, $\mu = M_{pl} N_{DL}^{-3/10} \sim 1 GeV \sim m_N$, where m_N is the nucleon mass. The Fig.6 strongly suggests that the degree of freedom of the universe (space-time) is given by : Eq.(8H) $\frac{\Lambda^4}{\mu^4} = \frac{\omega^4}{T^4} = N_{DL}^{6/5} \sim 10^{74} \sim (\frac{M_{pl}}{m_N})^4$.

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